Adaptive Partitioning for Chance-Constrained Problems with Finite Support

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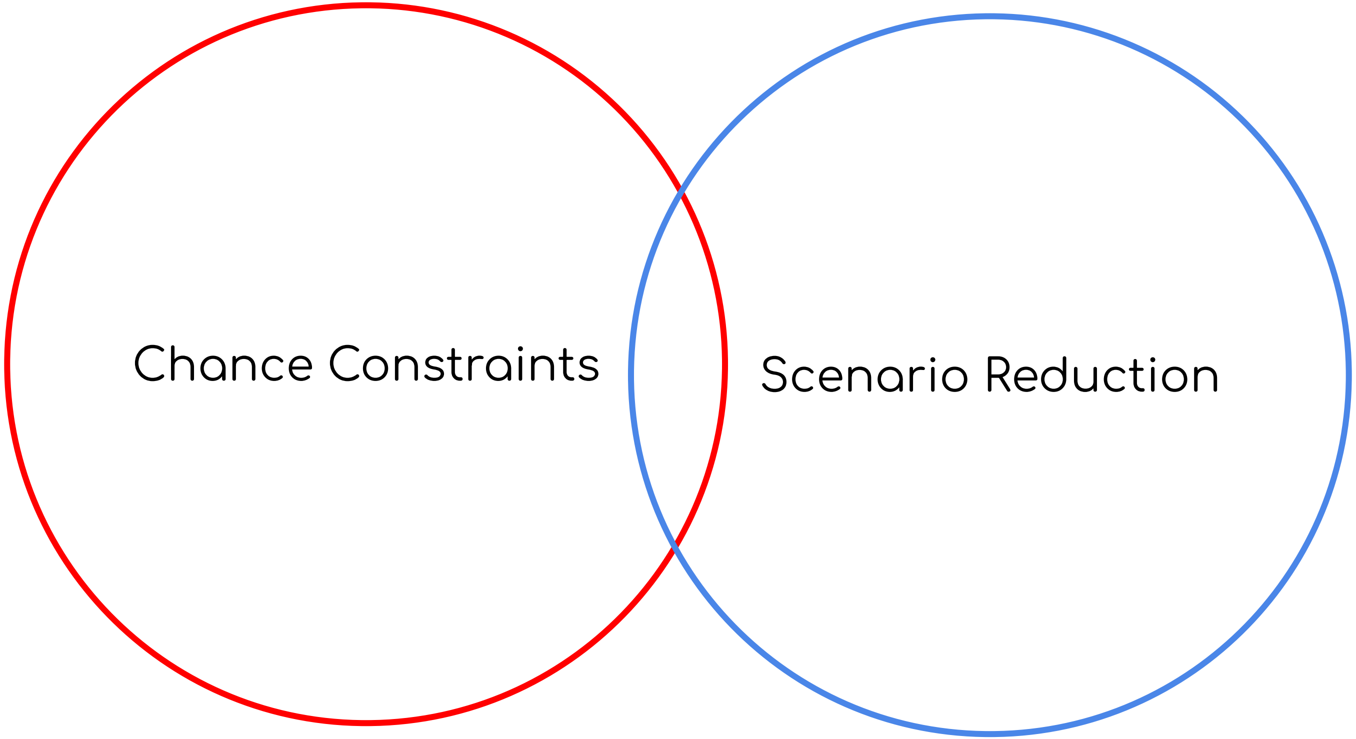
Introduction

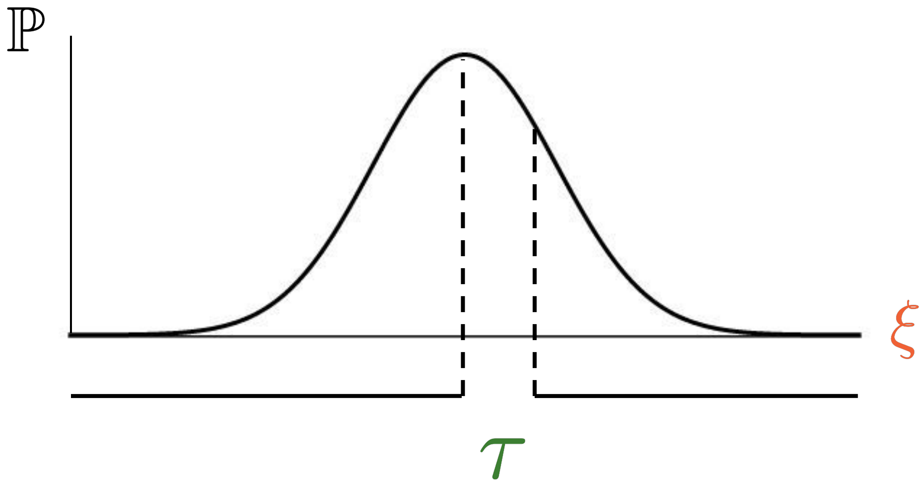


Chance-Constrained Stochastic Programs (CCSPs)

Model:

$$v^* = \min_{x \in \mathcal{X}} \quad f(x)$$
$$\text{s.t.} \quad \mathbb{P}_{\xi}[x \in X(\xi)] \geq 1 - \tau$$

Example:



Some Applications of CCSPs

Energy [Porras et al., 2023]:

$$\mathbb{P}(\text{network stability}) \geq 1 - \tau$$

Vehicle Routing [Errico et al., 2018]:

$$\mathbb{P}(\text{successful delivery}) \geq 1 - \tau$$

Finance [Cattaruzza et al., 2023]:

$$\mathbb{P}(\text{achieve desired return}) \geq 1 - \tau$$

CCSP with Finite Support

Assume

$$\xi \in \{\xi^s \in \mathbb{R}^d : \; s \in S\}$$

New Parameters & Variables

$$p_s \in [0,1], \; X^s = X(\xi^s) \qquad\qquad z_s \in \{0,1\}$$

Model

$$\begin{aligned} v^* = \min_{x \in \mathcal{X}} \quad & f(x) \\ \text{s.t.} \quad & \mathbb{P}_\xi[x \in X(\xi)] \geq 1 - \tau \end{aligned} \quad \longrightarrow \quad \begin{aligned} v^* = \min_{x \in \mathcal{X}} \quad & f(x) \\ \text{s.t.} \quad & z_s = \mathbb{1}(x \in X^s), \quad s \in S \\ & \sum_{s \in S} p_s z_s \geq 1 - \tau \\ & z_s \in \{0,1\}, \quad s \in S \end{aligned}$$

Let's Construct a Feasible Set

Model

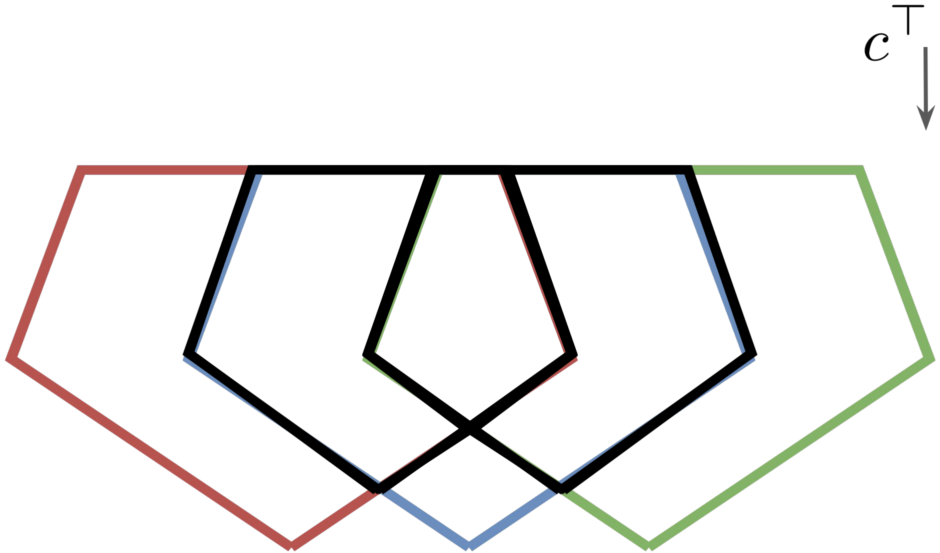
$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$

$$\text{s.t.} \quad z_s = \mathbb{1}(A^s x \geq b^s), \quad s \in \{1, 2, 3\}$$

$$\sum_{s \in \{1,2,3\}} z_s \geq 2$$

$$z_s \in \{0, 1\}$$

X-Space



→ Nonconvex region & $\binom{|S|}{\lfloor \tau |S| \rfloor}$ Operations

Related Literature

- Mixing Inequalities: [Kücükyavuz 2012, Luedtke, 2014, Abdi & Fukasawa, 2016]
- Quantile Cuts: [Qiu et al., 2014, Xie & Ahmed 2018]
- Big-M Tightening: [Qiu et al. 2014, Song et al. 2014, Porras et al. 2023]
- Relaxations: [Ahmed et al. 2017]
- Adaptive Partitioning Method: [Song et al. 2015, Espinoza et al. 2014]

Our Approach - Partitioned CCSP (P-CCSP)

Scenarios → Partitions

$$S = \{1, 2, 3, 4, 5\}$$

$$P = \{\{1, 4\}, \{2, 3, 5\}\}$$

Model [Ahmed et al., 2018]

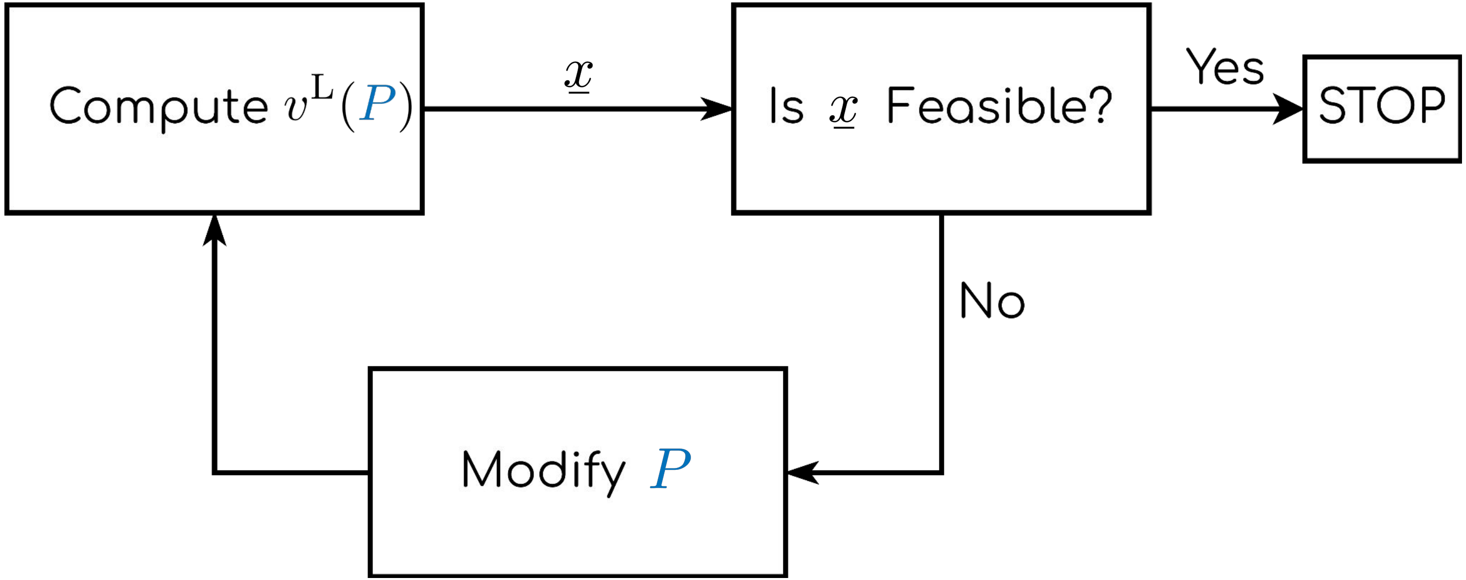
$$\begin{aligned} v^* = \min_{x \in \mathcal{X}} \quad & f(x) \\ \text{s.t.} \quad & z_s = \mathbb{1}(x \in X^s), \quad s \in S \\ & \sum_{s \in S} p_s z_s \geq 1 - \tau \\ & z_s \in \{0, 1\}, \quad s \in S \end{aligned}$$

$$\geq$$

$$\begin{aligned} v^{\mathrm{L}}(P) = \min_{x \in \mathcal{X}} \quad & f(x) \\ \text{s.t.} \quad & z_p = \mathbb{1}(x \in X^p), \quad p \in P \\ & \sum_{p \in P} p_p z_p \geq 1 - \varepsilon \\ & z_p \in \{0, 1\}, \quad p \in P \end{aligned}$$

→ Same constraints but **less variables**

Adaptive Partitioning I



→ Terminates finitely with the **optimal** solution if $|P^{j+1}| > |P^j|$

Modification by Refinement

Partition

$$P = \{\{1, 4\}, \{2, 3, 5\}\}$$

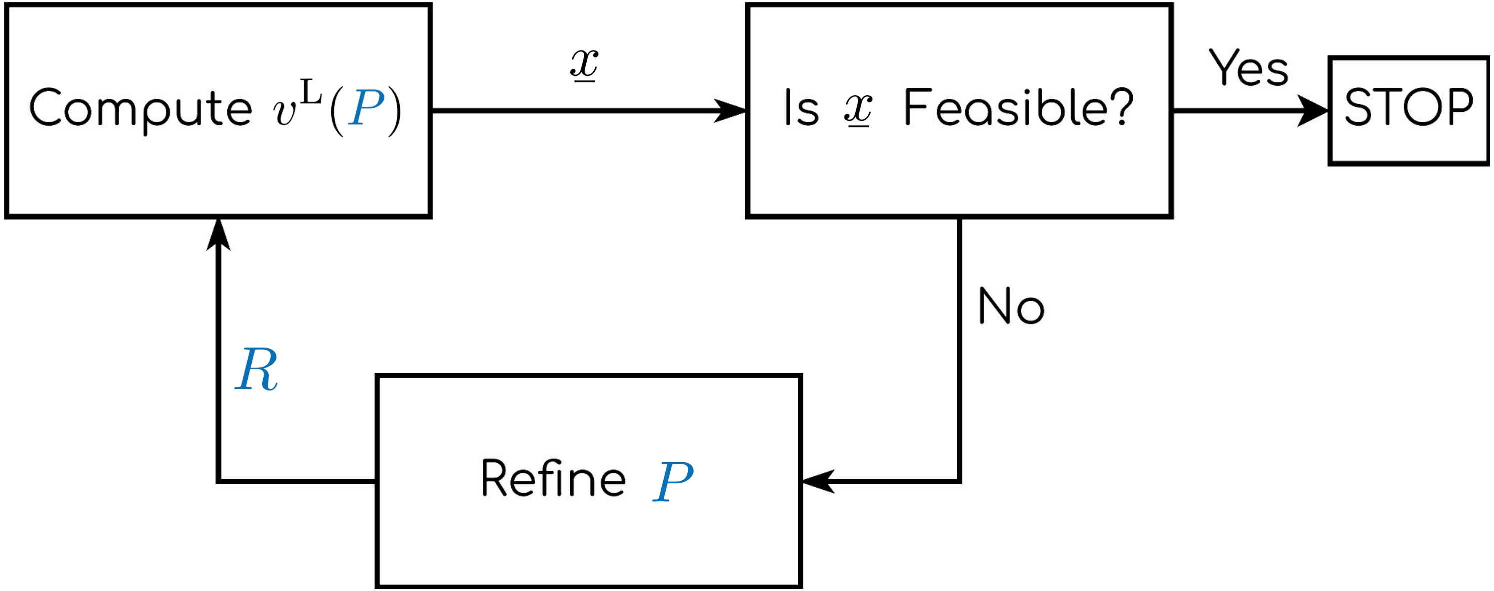
Refinement

$$R = \{\{1, 4\}, \{2, 5\}, \{3\}\}$$

Proposition. Model (P-CCSP) with P is a **relaxation** of Model (P-CCSP) with R , i.e.,

$$v^* \geq v^L(R) \geq v^L(P).$$

Adaptive Partitioning II



→ Create R so that $\underline{x}$ is not feasible for (P-CCSP) with R

What can we prove?

Theorem. If the optimal solution $\underline{x}$ of (P-CCSP) with P is **unique**, then we can create R of **minimal size** such that

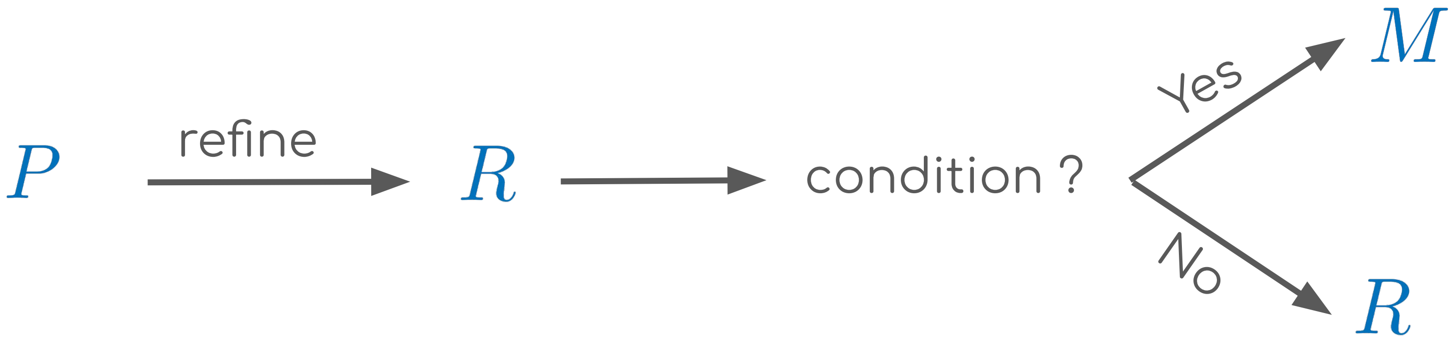
$$v^{\mathrm{L}}(R) > v^{\mathrm{L}}(P).$$

→ The size of R grows in each iteration 😭

→ Instead, can we create a merger M of P with the same ideas?

→ Impossible, P is a refinement of M. So, $v^{\mathrm{L}}(P) \geq v^{\mathrm{L}}(M)$ 😭

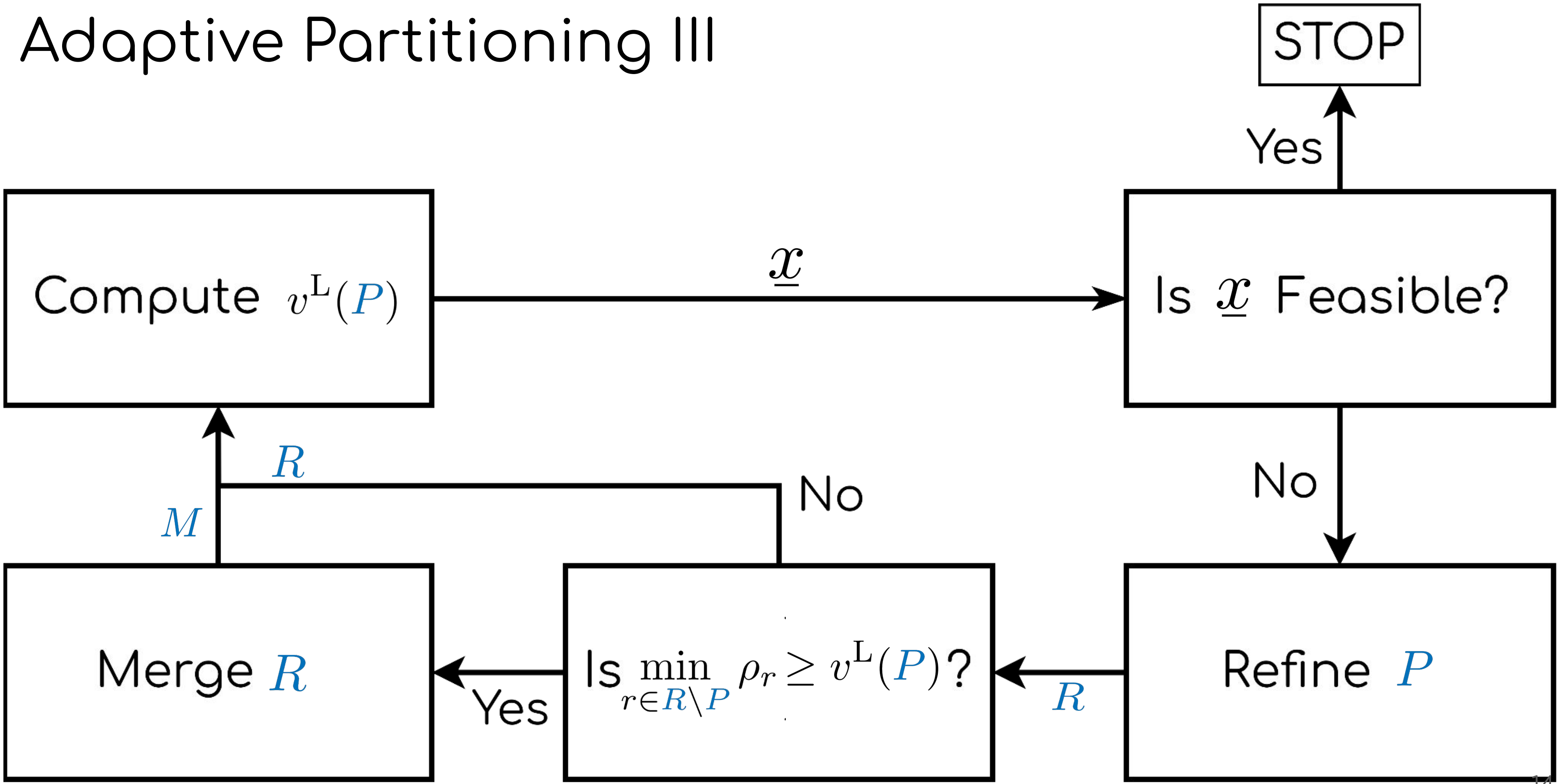
A-posteriori Merging



Theorem. If the optimal solution $\underline{x}$ of (P-CCSP) with P is **unique**, then we can create M such that $|P| = |M|$ and

$$v^{\mathrm{L}}(M) > v^{\mathrm{L}}(P).$$

Adaptive Partitioning III



Additions

- How to **project** $\underline{x}$ in the feasible space of CCSP?

- How to **refine** in practice?

- How to **merge** in practice?

- How to create **first** P such that $v^{\mathrm{L}}(P)$ is tight?

Addition: Strong Partitions

Proposition [Ahmed et al., 2018]. The **quantile bound** v_{Q} is a lower bound on the optimal objective of the CCSP, i.e.,

$$v^* \geq v_{\mathrm{Q}}$$

→ We can create P such that

$$v^* \geq v^{\mathrm{L}}(P) \geq v_{\mathrm{Q}}$$

Numerical Results

- Instances: Perturbed Multidimensional Knapsack [Song et al., 2014]

  → MIP with Big-M Constraints

  → Average over 5 Instances

- Variables: Continuous & Binary
- Time: 1 hour
- Benchmark/Comparison:

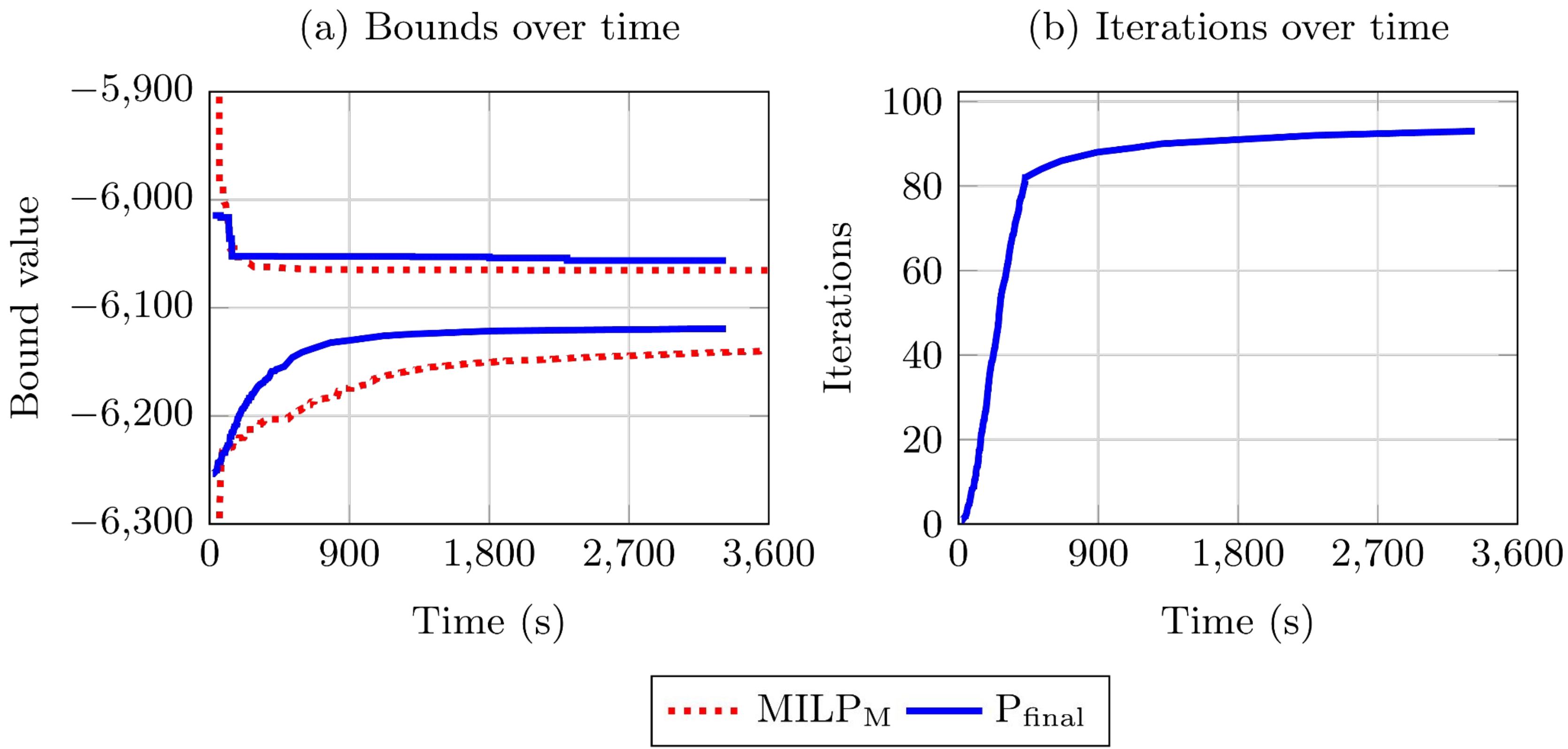
  → [Song et al., 2014]

  → [Belotti et al., 2016]

Numerical Results: Continuous Variables

			MILP$_M$		Adaptive P			
			Song Big-M	Belotti Big-M	P_{naive}		P_{final}	
Instance	τ	$\lvert S\rvert$	T_{avg}	T_{avg}	T_{avg}	It_{avg}	T_{avg}	It_{avg}
mk-20-10	0.1	500	**1310.13s**	0.26%(0)	0.95%(0)	30.0	0.24%(0)	46.0
		1000	**0.57%(0)**	1.68%(0)	1.97%(0)	17.0	0.69%(0)	77.0
		3000	1.64%(0)	6.01%(0)	3.10%(0)	13.0	**1.16%(0)**	189.0
		5000	2.06%(0)	5.17%(0)	2.88%(0)	11.0	**2.05%(0)**	120.0
	0.2	500	**0.47%(0)**	0.83%(0)	3.53%(0)	17.0	0.85%(0)	35.0
		1000	1.56%(0)	3.64%(0)	7.07%(0)	14.0	**1.52%(0)**	58.0
		3000	2.29%(0)	7.61%(0)	6.00%(0)	15.0	**1.79%(0)**	129.0
		5000	3.06%(0)	8.26%(0)	4.27%(0)	10.0	**1.96%(0)**	276.0
mk-40-30	0.1	500	**1.46%(0)**	5.50%(0)	2.68%(0)	11.0	2.97%(0)	58.0
		1000	**3.62%(0)**	15.09%(0)	5.13%(0)	7.0	6.03%(0)	27.0
		3000	-	24.57%(0)	12.16%(0)	3.0	**7.22%(0)**	11.0
		5000	-	24.29%(0)	15.70%(0)	2.0	**9.05%(0)**	5.0
	0.2	500	3.10%(0)	11.16%(0)	4.21%(0)	8.0	**2.34%(0)**	67.0
		1000	**6.89%(0)**	22.54%(0)	13.45%(0)	5.0	8.31%(0)	98.0
		3000	-	42.83%(0)	28.97%(0)	3.0	**10.66%(0)**	53.0
		5000	-	43.05%(0)	33.34%(0)	2.0	**12.38%(0)**	18.0

Numerical Results: mk-20-10, 0.2, 1000, continuous



Numerical Results: Binary Variables

			$MILP_M$		Adaptive P			
			Song Big-M	Belotti Big-M	P_{naive}		P_{final}	
Instance	τ	$\|S\|$	T_{avg}	T_{avg}	T_{avg}	It_{avg}	T_{avg}	It_{avg}
mk-10-10	0.1	500	10.39s	19.07s	21.70s	3.0	**6.25s**	1.0
		1000	**34.03s**	82.27s	132.77s	7.0	138.00s	7.0
		3000	231.59s	263.36s	401.06s	5.0	**206.13s**	2.0
		5000	606.85s	673.99s	641.46s	6.0	**443.65s**	4.0
	0.2	500	8.26s	12.15s	42.30s	6.0	**6.30s**	1.0
		1000	27.52s	38.64s	64.53s	4.0	**13.74s**	1.0
		3000	224.74s	267.23s	297.38s	7.0	**40.96s**	1.0
		5000	612.86s	477.06s	640.08s	7.0	**56.08s**	1.0

Conclusion

- We consider generic CCSPs with finite support

- A new method based on scenario reduction

- Results are promising, especially when $|S|$ is large

→ Combining Scenario Reduction & Optimization is promising!

→ Soon preprint available on mariusroland.gitlab.io