Exact and Heuristic Solution Techniques for Mixed-Integer Quantile Minimization Problems

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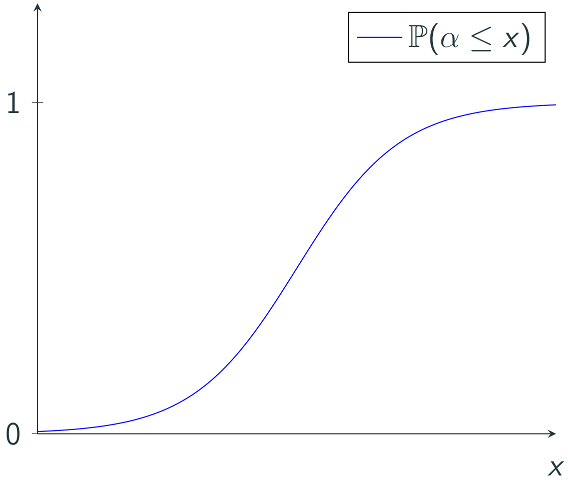
Quantile - VaR: Continuous Case

Let $\alpha \sim P_\alpha$ and $\tau \in [0, 1]$



Quantile - VaR: Discrete Case

Let $\alpha \sim P_\alpha$ and $\tau \in [0, 1]$, given a set of sorted samples α_i with probability p_i

$\mathbb{Q}_\tau[\alpha]$ is equal to

$$\min_k \ \alpha_k$$

such that

$$\tau \leq \sum_{i \leq k} p_i$$

Problem Statement

Given: $c_t \in \mathbb{R}^N, \; f : \mathbb{R} \to \mathbb{R}$

Find: $x \in X \subseteq \mathbb{R}^N$ such that

$$\min_{x} \quad \alpha \sum_{t \in \mathcal{T}} \mathbb{E}[c_t^\top x] + (1 - \alpha) \sum_{t \in \mathcal{T}} f\left(\mathbb{Q}_\tau[c_t^\top x]\right)$$

Our Focus:

- for all $t \in \mathcal{T}$ we sample a set of scenarios $\mathcal{S}_t$
- for all $s \in \mathcal{S}_t$ we have a cost $c_t^s \in \mathbb{R}^N$ and a probability $p_t^s \in [0, 1]$

Applications - Importance

- Risk measure (Benati and Rizzi 2007; Gaivoronski and Pflug 2005; Lin 2009; Mansini et al. 2003)
- Chance constraints (Song and Luedtke 2013; Song, Luedtke, and Küçükyavuz 2014; Tanner and Ntaimo 2010)
- ROADEF/EURO Challenge 2020: Maintenance Planning Problem[1]

[1] https://www.roadef.org/challenge/2020/en/index.php

Overview

Scenario Based Discrete Reformulation

Scenario Based Discrete Reformulation: Timestep

Given: cost $c^s \in \mathbb{R}^N$ and probability $p^s \in [0,1]$ for every scenario $s \in \mathcal{S}$

$$\mathbb{E}[c^\top x] = \sum_{s \in \mathcal{S}} p^s (c^s)^\top x$$

$$\begin{aligned} \mathbb{Q}_\tau[c^\top x] = \\ \arg\min_{q,y} \quad & q \\ \text{s.t.} \quad & q \geq (c^s)^\top x + M_t^s (y^s - 1), \quad s \in \mathcal{S} \\ & \sum_{s \in \mathcal{S}} y^s p^s \geq \tau \\ & y^s \in \{0,1\}, \quad s \in \mathcal{S} \end{aligned}$$

Scenario Based MILP

$$
\begin{aligned}
\min_{x,y,q} \quad & \alpha \sum_{t \in \mathcal{T}} \sum_{s \in \mathcal{S}_t} p_t^s (c_t^s)^\top x + (1-\alpha) \sum_{t \in \mathcal{T}} f\left(q_t\right) \\
\text{s.t.} \quad & q_t \geq (c_t^s)^\top x + M_t^s (y_t^s - 1), \quad t \in \mathcal{T}, s \in \mathcal{S}_t \\
& \sum_{s \in \mathcal{S}_t} y_t^s p_t^s \geq \tau, \quad t \in \mathcal{T} \\
& y_t^s \in \{0, 1\}, \quad t \in \mathcal{T}, s \in \mathcal{S}_t \\
& x \in X \subseteq \mathbb{R}^N
\end{aligned}
$$

$$
\begin{aligned}
\min_{x,y,q} \quad & \alpha \sum_{t \in \mathcal{T}} \sum_{s \in \mathcal{S}_t} p_t^s (c_t^s)^\top x + (1-\alpha) \sum_{t \in \mathcal{T}} f\left(q_t\right) \\
\text{s.t.} \quad & q_t \geq (c_t^s)^\top x + M_t^s (y_t^s - 1), \quad t \in \mathcal{T}, s \in \mathcal{S}_t
\end{aligned}
$$

Solution Methods

Valid Inequalities: Presentation

Given: $\bar{\mathcal{S}} \subseteq \mathcal{S}$ such that $p(\bar{\mathcal{S}}) < \tau$

VI1 (Kleinert et al. 2021):

$$\left(\tau - p(\bar{\mathcal{S}})\right) q \geq \sum_{i=1}^{n} \left(b_i(\emptyset) - c_i(\bar{\mathcal{S}})\right) x_i$$

$\Rightarrow$ Separation in P

VI2 (Qiu et al. 2014):

$$\left(\tau - p(\bar{\mathcal{S}})\right) q \geq \sum_{i=1}^{n} b_i(\bar{\mathcal{S}}) x_i$$

$\Rightarrow$ Separation NP-hard

$\Rightarrow$ Dominates **VI1**

Valid Inequalities: Summary

Separate **VI2** with **VI1**

Overlapping Alternating Direction Method: Idea

$$
\begin{aligned}
\min_{x,y,q} \quad & \sum_{s\in\mathcal{S}} p^s (c^s)^\top x + (1-\alpha) \sum_{t\in\mathcal{T}} f\left(q_t\right) \\
\text{s.t.} \quad & q_t \geq (c_t^s)^\top x + M_t^s (y_t^s - 1), \quad t \in \mathcal{T}, s \in \mathcal{S}_t, \\
& \sum_{s\in\mathcal{S}_t} y_t^s p_t^s \geq \tau, \quad t \in \mathcal{T}, \\
& y_t^s \in \{0,1\}, \quad t \in \mathcal{T}, s \in \mathcal{S}_t, \\
& x \in X \subseteq \mathbb{R}^N.
\end{aligned}
$$

Ideas:

- Decompose in smaller subproblems
- Fix x or y and solve over remaining variables

Adaptive Scenario Clustering Algorithm: Idea I

Observation 1:

Size of $\mathcal{S}_t$ affects $\mathbb{Q}_\tau[c_t^\top x]$ approximation but not $\mathbb{E}[c_t^\top x]$ approximation

Observation 2:

Given $x \in X \to q_t$ value defined by exactly one $s \in \mathcal{S}_t$

Idea:

Clustering $\mathcal{C}_t$ of $\mathcal{S}_t$

Problem:

How to cluster $\mathcal{S}_t$?

Adaptive Scenario Clustering Algorithm: Idea II

Adaptive Scenario Clustering Algorithm: Global Optimality

Given: $\mathcal{C}_t$ a partition of $\mathcal{S}_t$ in $k_t \leq |\mathcal{S}_t|$ non-empty clusters

Average (ASC):

$$(c_t^\gamma)_i = \frac{1}{|\gamma|} \sum_{s \in \gamma} (c_t^s)_i, \quad p_t^\gamma = \sum_{s \in \gamma} p_t^s, \quad \gamma \in \mathcal{C}_t$$

Min (MSC):

$$(c_t^\gamma)_i = \min \left\{ (c_t^s)_i \colon s \in \gamma \right\}, \quad p_t^\gamma = \sum_{s \in \gamma} p_t^s, \quad \gamma \in \mathcal{C}_t$$

Bounds:

$$v_{\text{MSC}}(x^*_{\text{MSC}}) \leq v(x^*) \leq \min\{v(x^*_{\text{ASC}}), v(x^*_{\text{MSC}})\}$$

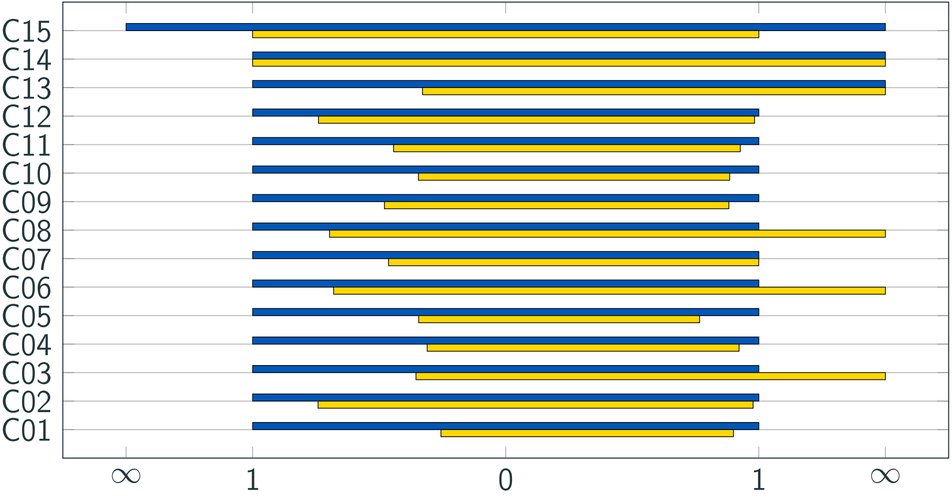
Adaptive Scenario Clustering Algorithm: Idea II

Numerical Results

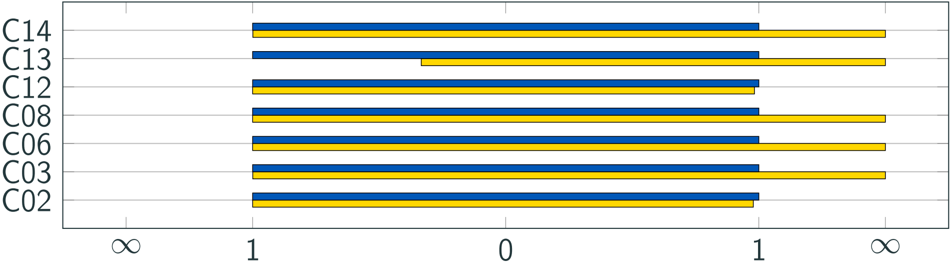
Software & Hardware Setup

- ROADEF/EURO Challenge 2020 semi-final instances
- Time limit: 90 min
- MILP solver: Gurobi 9.1.0.
- All techniques implemented in C++14 and compiled with g++ version 9.3.0
- AMD Opteron 6176 SE processor with 12 cores at 2.30 GHz and 64 GB RAM

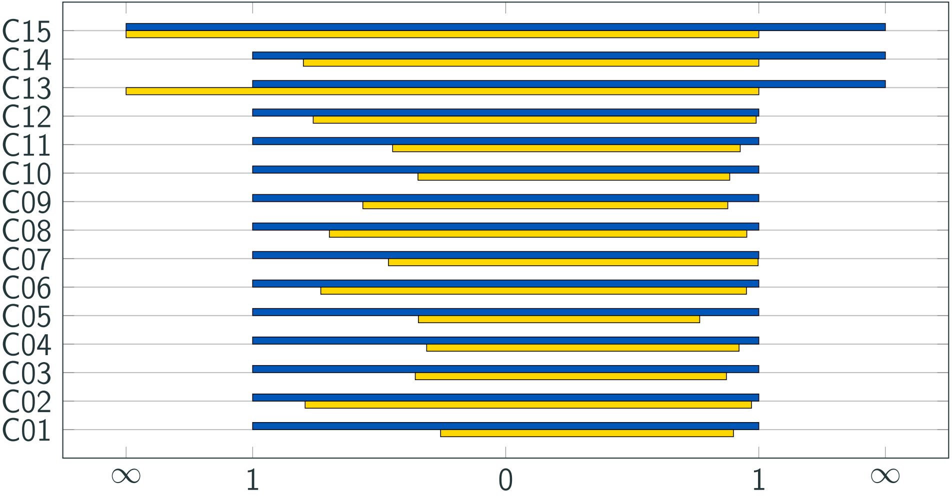
MILP (blue) vs. $MILP_{VI}$ (yellow)



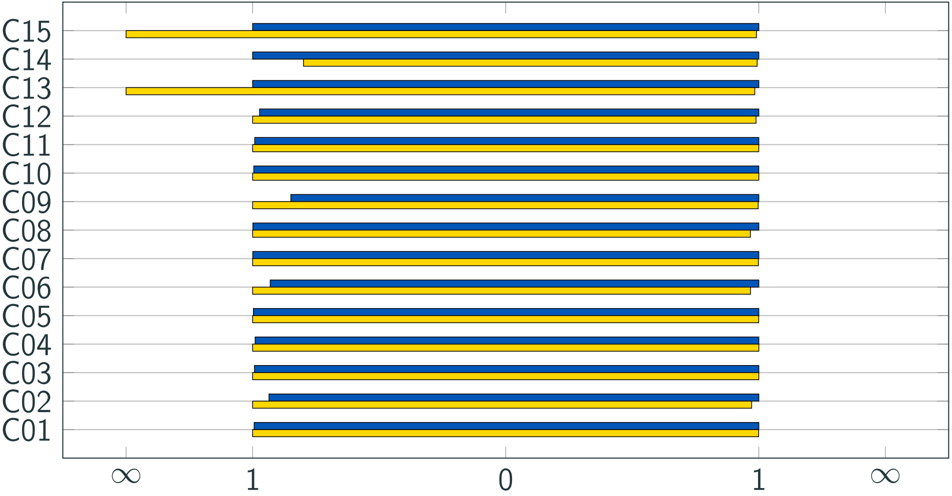
$\mathrm{MILP}_{\mathrm{VI}}^{\mathrm{OADM}}$ (blue) vs. $\mathrm{MILP}_{\mathrm{VI}}$ (yellow)



MILP (blue) vs. ASCA (yellow)



$\text{MILP}_{\text{VI}}^{\text{OADM}}$ (blue) vs. ASCA (yellow)



That's It !

Summary:

- Scenario Based Discrete Reformulation
- Valid Inequalities, OADM, ASCA
- Significantly outperform MILP solvers

Next Steps:

- Chance Constraints